TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 224, Number 2, 1976

A MINIMAX FORMULA FOR DUAL B^*-ALGEBRAS

BY

PAK-KEN WONG

ABSTRACT. Let A be a dual B^*-algebra. We give a minimax formula for the positive elements in A. By using this formula and some of its consequent results, we introduce and study the symmetric norms and symmetrically-normed ideals in A.

1. **Introduction.** Let H be a (complex) Hilbert space and $LC(H)$ the algebra of all compact operators on H. Then $LC(H)$ is a simple dual B^*-algebra and every simple dual B^*-algebra is of this form. The minimax formula for the positive elements in $LC(H)$ is well known and has many applications (see [2, p. 908, Theorem 3] and [3, p. 25, Theorem]). In this paper, we present a generalization of this formula to the positive elements in an arbitrary dual B^*-algebra. Let a be a positive element in a dual B^*-algebra A and E the set of all Hermitian minimal idempotents in A. We show that the singular values $s_n(a)$ of a can be calculated by the following equations

$$s_1(a) = \max\{\|eae\|: e \in E\},$$
$$s_{n+1}(a) = \min_{f_1,\ldots,f_n \in E} \max\{\|eae\|: e \in E, ef_i = 0, i = 1, 2, \ldots, n\}.$$

After establishing this formula, we give some applications. Let $R_0 = (0)$ and let R_n be the set of all elements $x = \Sigma_{j=1}^n x_j f_j$ in A, where $x_j \in A$ and $f_j \in E$ such that $f_i f_j = 0$ $(i \neq j)$. We show that, for any element a in A, its singular values are given by

$$s_{n+1}(a) = \min\{\|a - b\|: b \in R_n\} \quad (n = 0, 1, 2, \ldots).$$

We also obtain that, for all elements a, b in A,

$$\sum_{n=1}^{k} s_n(a + b) \leq \sum_{n=1}^{k} s_n(a) + \sum_{n=1}^{k} s_n(b) \quad (k = 1, 2, \ldots)$$

and

Received by the editors October 1, 1975.

AMS (MOS) subject classifications (1970). Primary 46C05; Secondary 46K99.

Key words and phrases. Dual algebra, A^*-algebra, B^*-algebra, Hermitian minimal idempotent, symmetrically-normed ideal, symmetric norming function.

$$\prod_{n=1}^{k} s_n(ab) \leqslant \prod_{n=1}^{k} s_n(a) \prod_{n=1}^{k} s_n(b) \qquad (k = 1, 2, \ldots).$$

These inequalities were obtained by K. Fan and A. Horn for compact operators on a Hilbert space (see [3, p. 48, Lemma 4.2]).

The properties of symmetric norming functions and symmetric norm (uniform crossnorm) in $LC(H)$ are well known and have been studied by many mathematicians (e.g. see [3] and [8]). In this paper, we introduce the concepts of symmetric norm and symmetrically-normed ideals for A. Let S_A be the socle of A. We show that the class of all s.n. functions and the class of all symmetric norms on S_A generate each other.

Let Φ be an s.n. function, A_Φ the s.n. ideal generated by Φ and $A_\Phi^{(0)}$ the closure of S_A in A_Φ. We prove that $A_\Phi^{(0)}$ is a dual A^*-algebra and the conjugate space of $A_\Phi^{(0)}$ can be identified with A_{Φ^*}, where Φ^* denotes the s.n. function adjoint to Φ. The formulas established above are useful in the proof of these results. We also remark that for the case $A = LC(H)$, all these results were known.

In this paper, our approach is elementary and basically algebraic. The technique in the proof of the minimax formula is quite different from that used in [2] and [3].

2. **Notation and preliminaries.** Definitions not explicitly given are taken from Rickart's book [6].

For any set S in a Banach algebra A, let $l_A(S)$ and $r_A(S)$ denote the left and right annihilators of S in A, respectively. Then A is called a dual algebra, if for every closed right ideal R and every closed left ideal I, we have $r_A(l_A(R)) = R$ and $l_A(r_A(I)) = I$. See [5] and [6] for some of its properties.

An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semisimple, this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A.

Let A be a Banach algebra. A bounded linear operator T on A is called a right centralizer if $T(xy) = (Tx)y$ for all x, y in A. For each a in A, the operator L_a: $x \longrightarrow ax$ ($x \in A$) is a right centralizer on A.

In this paper, all algebras and linear spaces under consideration are over the field C of complex numbers.

Notation. In this paper, A will denote a dual B^*-algebra with norm $\|\cdot\|$.

We shall use, without explicitly mentioning, the following fact: For any orthogonal family $\{e_\alpha\}$ of Hermitian minimal idempotents of A, $\Sigma_\alpha e_\alpha x$ is summable in A, and especially when $\{e_\alpha\}$ is a maximal family, $x = \Sigma_\alpha e_\alpha x$ for all x in A (see [5, p. 30, Theorem 16] and [10, p. 442, Theorem 5.2]).

Let b be a normal element in A and $\mathrm{Sp}_A(b)$ the spectrum of b in A. Then it is well known that $\mathrm{Sp}_A(b)$ is either finite or countable, and has no nonzero limit points (see [1, p. 502, Corollary]). Let $\{e_\alpha\}$ be a maximal orthogonal family of Hermitian minimal idempotents in A such that $e_\alpha b = be_\alpha$ for all α. By [6, p. 111, Theorem (3.1.6)], each $\lambda_\alpha \in \mathrm{Sp}_A(b)$, where $\lambda_\alpha e_\alpha = be_\alpha = e_\alpha be_\alpha$, and $\mathrm{Sp}_A(b) - (0) \subset \{\lambda_\alpha\}$. Let λ be a nonzero number in $\mathrm{Sp}_A(b)$. If the set $\{\lambda_\alpha\colon \lambda_\alpha = \lambda\}$ has k_λ elements, then the number k_λ is finite and independent of the choice of $\{e_\alpha\}$ (see the proof of Lemma 2.3 in [11]). We call k_λ the multiplicity of λ. Let $\{\lambda_n\} = \{\lambda_\alpha\colon \lambda_\alpha \neq 0\}$. Then $\{\lambda_n\}$ is countable and $b = \Sigma_\alpha e_\alpha b = \Sigma_n \lambda_n e_n$, where $e_n \in \{e_\alpha\}$ with $e_n b = \lambda_n e_n$ (see [11]). It is clear that $\{\lambda_n\}$ is independent of $\{e_\alpha\}$ and, if $\lambda_\alpha \neq \lambda_n$ for all n, then $\lambda_\alpha = 0$. The numbers λ_n are called the eigenvalues of b.

Now suppose a is a nonzero element in A. Then a^*a is a positive element and so each nonzero number in $\mathrm{Sp}_A(a^*a)$ is positive. Let $\{\lambda_n\}$ be the eigenvalues of a^*a, arranged in decreasing order and repeated according to multiplicity. Then $\lambda_n \geqslant 0$ and $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$. Let $\{e_\alpha\}$ be a maximal orthogonal family of Hermitian minimal idempotents of A such that $e_\alpha a^*a = a^*ae_\alpha$ for all α. Then $a^*a = \Sigma_\alpha a^*ae_\alpha = \Sigma_n \lambda_n e_n$ and

$$a = \sum_n ae_n, \tag{2.1}$$

where $e_n \in \{e_\alpha\}$ with $\lambda_n e_n = a^*ae_n$ (see [11]). Put $s_n(a) = \sqrt{\lambda_n}$.

DEFINITION. The number $s_n(a)$ is called the nth singular value of the element a in A.

REMARK. Since $\lambda_1 = \|a^*a\| = \|a\|^2$, $s_1(a) = \|a\|$.

Let us put

$$[a] = \sum_n s_n(a)e_n. \tag{2.2}$$

Then $[a] = [a]^* = (a^*a)^{1/2}$ (see [11]). Define two mappings W and W^* on A into itself by

$$Wx = \sum_n (s_n(a))^{-1}ae_n x \qquad (x \in A) \tag{2.3}$$

and

$$W^*x = \sum_n (s_n(a))^{-1}e_n a^*x \qquad (x \in A). \tag{2.4}$$

Then we can show that W and W^* are right centralizers on A with $\|W\| = \|W^*\| = 1$, $W[a] = a$ and $W^*a = [a]$ (see [11]). We shall refer to the operator W as the partial isometry asssociated with a.

3. **A minimax formula for A.** Let A be a dual B^*-algebra with norm $\|\cdot\|$ and E the set of all Hermitian minimal idempotents in A. Then by [6, p. 98,

Lemma (2.8.6)] and [6, p. 261, Lemma (4.10.1)], every nonzero left or right ideal of A contains some element of E.

LEMMA 3.1. *Let M be a maximal modular right ideal and R a nonzero right ideal of A such that $M \cap R = (0)$. Then R is a minimal right ideal of A.*

PROOF. By [6, p. 98, Lemma (2.8.6)], R contains a minimal right ideal I. Since M is maximal, it follows that $M \oplus R = M \oplus I = A$. Therefore $R = I$.

LEMMA 3.2. *Let $M_1, M_2, \ldots, M_n$ be maximal modular right ideals and $e_1, e_2, \ldots, e_{n+1}$ any mutually orthogonal Hermitian minimal idempotents in A. Then*

$$M_1 \cap M_2 \cap \cdots \cap M_n \cap (e_1 + e_2 + \cdots + e_{n+1})A \neq (0). \tag{3.1}$$

PROOF. We use induction. If $k = 1$, the lemma follows easily from Lemma 3.1. Now suppose that the lemma is true for $k = n - 1$. Since

$$\begin{aligned}(e_{m+1} + e_{m+2} + \cdots + e_{m+p})\\ = (e_1 + e_2 + \cdots + e_{n+1})(e_{m+1} + e_{m+2} + \cdots + e_{m+p})\end{aligned} \tag{3.2}$$

with $1 \leq m + p \leq n + 1$, we see that $(e_{m+1} + e_{m+2} + \cdots + e_{m+p})A \subset (e_1 + e_2 + \cdots + e_{n+1})A$. If there exists some $e_i \in M_1 \cap M_2 \cap \cdots \cap M_n$ $(1 \leq i \leq n)$, then (3.1) clearly holds. Therefore, without loss of generality, we may assume that $e_1 \notin M_1$. It follows from Lemma 3.1 that $M_1 \cap (e_1 + e_j)A \neq 0$ $(j = 2, 3, \ldots, n + 1)$. Hence for each j, there exists a Hermitian minimal idempotent $h_j \in M_1 \cap (e_1 + e_j)A$; clearly $h_j = (e_1 + e_j)h_j$. We claim that $e_jh_j \neq 0$. In fact, if $e_jh_j = 0$, then $h_j = e_1h_j \in e_1A$. Therefore, by [6, p. 261, Lemma (4.10.1)], $e_1 = h_j \in M_1$; a contradiction. Hence $e_jh_j \neq 0$. If there exists some $2 \leq p \leq n + 1$ such that $h_pA \cap \Sigma_{j\neq p} h_jA \neq (0)$, then $h_p \in h_pA \subset \Sigma_{j\neq p} h_jA$. Hence $h_p = \Sigma_{j\neq p} h_jx_j$ with $x_j \in A$. Then $e_ph_p = \Sigma_{j\neq p} e_p(e_1 + e_j)h_jx_j = 0$, which is a contradiction. Consequently, $h_2A + h_3A + \cdots + h_{n+1}A$ is a direct sum. Therefore by the proof of [1, p. 497, Theorem 2.2], we can find an orthogonal family $\{f_2, f_3, \ldots, f_{n+1}\}$ of Hermitian minimal idempotents contained in $h_2A + h_3A + \cdots + h_{n+1}A$. Hence by induction hypothesis, $M_2 \cap M_3 \cap \cdots \cap M_n \cap (f_2 + f_3 + \cdots + f_{n+1})A \neq (0)$. Since by (3.2), $h_j \in (e_1 + e_j)A \subset (e_1 + e_2 + \cdots + e_{n+1})A$, it follows easily that $(f_2 + f_3 + \cdots + f_{n+1})A \subset M_1 \cap (e_1 + e_2 + \cdots + e_{n+1})A$, and so (3.1) holds. This completes the proof.

LEMMA 3.3. *Let a be a positive element and e any Hermitian minimal idempotent in A. Then*

(i) *eae is positive.*

(ii) *If $\lambda e = eae$, then $\lambda = \|eae\|$.*

PROOF. (i). Write $a = h^*h$, with $h \in A$. Then $eae = (he)^*(he)$ is positive.

(ii). It follows easily from [6, p. 261, Theorem (4.10.3)] that $\lambda \geqslant 0$. Therefore $\lambda = \|eae\|$.

We now have the main result of this section.

THEOREM 3.4 (MINIMAX FORMULA). *Let a be a positive element in a dual B^*-algebra A with singular values $\{s_1(a), s_2(a), \ldots\}$ and E the set of all Hermitian minimal idempotents in A. Then for $n = 1, 2, \ldots$ we have*

$$(3.3)\qquad \begin{aligned} s_1(a) &= \max\{\|eae\|: e \in E\} \\ s_1(a) &= \min_{f_1 \in E} \max\{\|eae\|: e \in E, f_1 e = 0\} \\ &\;\;\vdots \\ s_{n+1}(a) &= \min_{f_1,\ldots,f_n \in E} \max\{\|eae\|: e \in E, ef_i = 0, i = 1, 2, \ldots, n\}. \end{aligned}$$

PROOF. We write $s_n = s_n(a)$ and $a = \Sigma_n s_n e_n$ (see (2.2)). Since for all $e \in E$, $\|eae\| \leqslant \|a\| = s_1$ and $s_1 = \|e_1 a e_1\|$, it follows that $s_1 = \max\{\|eae\|: e \in E\}$. Now let $f_1, f_2, \ldots, f_n$ be any elements in E. Put $M_k = (1 - f_k)A$ $(k = 1, 2, \ldots, n)$. Then by Lemma 3.2, there exists some $h \in E$ such that $h \in M_1 \cap M_2 \cap \cdots \cap M_n \cap (e_1 + e_2 + \cdots + e_{n+1})A$. Since $h \in M_k$, it follows that $f_k h = 0$ $(k = 1, 2, \ldots, n)$. Also $h = (e_1 + e_2 + \cdots + e_{n+1})h$. Write $hah = th$ with $t = \|hah\|$ (Lemma 3.3). Then

$$\begin{aligned} (t - s_{n+1})h &= ha(e_1 + e_2 + \cdots + e_{n+1})h - s_{n+1}h \\ &= h(s_1e_1 + s_2e_2 + \cdots + s_{n+1}e_{n+1})h \\ &\quad - s_{n+1}h(e_1 + e_2 + \cdots + e_{n+1})h \\ &= (s_1 - s_{n+1})he_1h + (s_2 - s_{n+1})he_2h + \cdots + (s_1 - s_{n+1})he_nh. \end{aligned}$$

Hence it follows from Lemma 3.3 and [6, p. 232, Lemma (4.7.4)] that $(t - s_{n+1})h$ is positive and so $t = \|hah\| \geqslant s_{n+1}$. Since $hf_k = 0$, we have

$$(3.4)\qquad s_{n+1} \leqslant \max\{\|eae\|: e \in E, ef_k = 0, k = 1, 2, \ldots, n\}.$$

Suppose $e \in E$ with $ee_k = 0$ $(k = 1, 2, \ldots, n)$. Then

$$\begin{aligned} \|eae\| &= \left\| e\left(\sum_{k=n+1}^{\infty} s_k e_k\right) e \right\| \leqslant \left\| \sum_{k=n+1}^{\infty} s_k e_k \right\| \\ &= \sup\{\|s_k e_k\|: k = n+1, n+2, \ldots\} = s_{n+1}. \end{aligned}$$

Also $\|e_{n+1}ae_{n+1}\| = s_{n+1}$. Consequently,

(3.5) $$s_{n+1} = \max\{\|eae\|: e \in E,\ ee_k = 0,\ k = 1, 2, \ldots, n\}.$$

Combining (3.4) and (3.5), we get (3.3) and this completes the proof.

REMARK. Let H be a Hilbert space and $LC(H)$ the algebra of all compact operators on H. It is well known that $LC(H)$ is a simple dual B^*-algebra. For each Hermitian minimal idempotent e in $LC(H)$, we can write $e = (x \otimes x)/(x, x)$ for some nonzero element x in H. Then for any Hermitian element T in $LC(H)$, $eTe = (Tx, x)e/(x, x)$. Therefore $\|eTe\| = |(Tx, x)|/(x, x)$. Let $f = (y \otimes y)/(y, y)$ be any Hermitian minimal idempotent in $LC(H)$. Then it is easy to see that $ef = 0$ if and only if $(x, y) = 0$. Hence, Theorem 3.4 is a generalization of [2, p. 908, Theorem 3].

The following corollaries are useful in the next section. They are known for the algebra $LC(H)$.

COROLLARY 3.5. *Let a be an element in a dual B^*-algebra A. Then the singular values $s_n(a)$ of a are given by*

$$s_1(a) = \max\{\|ae\|: e \in E\}$$

$$\vdots$$

$$s_{n+1}(a) = \min_{f_1, \ldots, f_n \in E} \max\{\|ae\|: e \in E,\ ef_i = 0,\ i = 1, 2, \ldots, n\}.$$

PROOF. This follows from Theorem 3.4 and the fact that $\|ae\|^2 = \|ea^*ae\| = \|e[a]^2e\|$.

COROLLARY 3.6. *Let $a \in A$ and T a bounded linear operator on A. If $s_n(Ta)$ are the singular values of Ta, then we have*

$$s_n(Ta) \leqslant \|T\| s_n(a) \qquad (n = 1, 2, \ldots).$$

PROOF. Since $\|Tae\| \leqslant \|T\|\ \|ae\|$, Corollary 3.6 follows easily from Corollary 3.5.

COROLLARY 3.7. *Let a and b be positive elements in A. If $a - b$ is a positive element, then $s_n(a) \geqslant s_n(b)$ $(n = 1, 2, \ldots)$.*

PROOF. Let $e \in E$. Then $e(a - b)e$ is positive. Hence $\|eae\| \geqslant \|ebe\|$. Therefore by the minimax formula, $s_n(a) \geqslant s_n(b)$.

4. Some applications of the minimax formula. In this section, by using the minimax formula, we shall generalize some known results for compact operators on Hilbert spaces. As before, A will be a dual B^*-algebra with norm $\|\cdot\|$ and E the set of all Hermitian minimal idempotents of A.

Let R_n be the set of all elements x in A such that $x = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n$, where $x_1, x_2, \ldots, x_n \in A$ and $f_1, f_2, \ldots, f_n$ are mutually orthogonal Hermitian minimal idempotents in A ($n = 1, 2, \ldots$). Put $R_0 = (0)$. Note that x_k ($k = 1, 2, \ldots, n$) can be zero and so $R_{n-1} \subset R_n$.

THEOREM 4.1. *Let a be an element in a dual B^*-algebra A. Then for $n =$ $0, 1, 2, \ldots$, we have*

$$s_{n+1}(a) = \min\{\|a - b\|: b \in R_n\}. \tag{4.1}$$

PROOF. If $n = 0$, then (4.1) reduces to $s_1(a) = \|a\|$. Suppose $n \geq 1$. Let $b = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n \in R_n$. If $e \in E$ and $ef_1 = ef_2 = \cdots = ef_n = 0$, then $\|ae\| = \|(a - b)e\| \leq \|a - b\|$. Hence it follows from Corollary 3.5 that

$$s_{n+1}(a) \leq \|a - b\| \qquad (n = 1, 2, \ldots). \tag{4.2}$$

Write $[a] = \sum_{k=1}^{\infty} s_k(a)e_k$ and $a = \sum_{k=1}^{\infty} ae_k$ (see (2.1) and (2.2)). Put $a_n = \sum_{k=1}^{n} ae_k$. Then $a_n \in R_n$ and

$$\begin{aligned} \|a - a_n\| &= \left\| \sum_{k=n+1}^{\infty} ae_k \right\| = \left\| \sum_{k=n+1}^{\infty} e_k[a]^2 e_k \right\|^{1/2} \\ &= \left\| \sum_{k=n+1}^{\infty} s_k^2(a)e_k \right\|^{1/2} = s_{n+1}(a). \end{aligned} \tag{4.3}$$

Now (4.1) follows immediately from (4.2) and (4.3) and this completes the proof.

REMARK. Theorem 4.1 is similar to [3, p. 28, Theorem 2.1].

The following result was obtained by K. Fan for compact operators (see [3, p. 29, Corollary 2.2]).

COROLLARY 4.2. *Let $a, b \in A$. Then the following statements are true for $m, n = 1, 2, \ldots$.*

(i) $s_{m+n-1}(a + b) \leq s_m(a) + s_n(b)$.

(ii) $|s_n(a) - s_n(b)| \leq \|a - b\|$.

(iii) $s_{m+n-1}(ab) \leq s_m(a)s_n(b)$.

PROOF. Let $u \in R_{m-1}$ and $v \in R_{n-1}$ be such that $s_m(a) = \|a - u\|$ and $s_n(b) = \|b - v\|$ (see the proof of Theorem 4.1).

(i) Since $u + v \in R_{m+n-2}$ (see [1, p. 497]), by Theorem 4.1, we have

$$s_{m+n-1}(a+b) \leqslant \|(a+b)-(u+v)\| \leqslant s_m(a)+s_n(b).$$

(ii) This follows immediately from (i).

(iii) Let $w = b - v$ and write $u = x_1e_1 + x_2e_2 + \cdots + x_{m-1}e_{m-1}$, where $e_1, e_2, \ldots, e_{m-1}$ are mutually orthogonal elements in E. We claim that $uw \in R_{m-1}$. In fact, if $e_iw \neq 0$, then by [6, p. 45, Lemma (2.18)], Ae_iw is a minimal left ideal. Consequently, we can write $Ae_1w + Ae_2w + \cdots + Ae_{m-1}w = A(f_1 + f_2 + \cdots + f_k)$, where $1 \leqslant k \leqslant m-1$ and $f_1, f_2, \ldots, f_k$ are mutually orthogonal elements in E (see [1, p. 497]). Therefore $uw \in R_{m-1}$. Since $(a-u)(b-v) = ab - av - uw$ and $av + uw \in R_{m+n-2}$, it follows from Theorem 4.1 that

$$\begin{aligned} s_{m+n-1}(ab) &\leqslant \|ab-(av+uw)\| \leqslant \|a-u\|\,\|b-v\| \\ &= s_m(a)s_n(b). \end{aligned}$$

This completes the proof.

By using Corollary 4.2 and the proof of [3, p. 32, Theorem 2.3], we have

COROLLARY 4.3. *Suppose $a, b \in A$ and $r > 0$. If $\lim_{n\to\infty} n^r s_n(a) = t$ and $\lim_{n\to\infty} n^r s_n(b) = 0$, then $\lim_{n\to\infty} n^r s_n(a+b) = t$.*

LEMMA 4.4. *Let $a \in A(f_1 + f_2 + \cdots + f_n)$, where $f_1, f_2, \ldots, f_n$ are mutually orthogonal Hermitian minimal idempotents in A. Then $s_{n+1}(a) = s_{n+2}(a) = \cdots = 0$.*

PROOF. Suppose this is not so. Then we can write $a^*a = \sum_{j=1}^k s_j^2(a)e_j$ with $n+1 \leqslant k \leqslant \infty$. Since $e_ja^*a = s_j^2(a)e_j$ and $s_j(a) \neq 0$, it follows that $e_j \in A(f_1 + f_2 + \cdots + f_n)$ $(j = 1, 2, \ldots, k)$, which is a contradiction (see [1, p. 497]). Hence the lemma is true.

LEMMA 4.5. *Let $a \in A$ and $f_1, f_2, \ldots, f_k$ any mutually orthogonal Hermitian minimal idempotents in A. Then*

$$\sum_{n=1}^{k} \|f_naf_n\| \leqslant \sum_{n=1}^{k} s_n(a) \qquad (k = 1, 2, \ldots). \tag{4.4}$$

PROOF. Let $\{f_\beta\}$ be any maximal orthogonal family of Hermitian minimal idempotents containing $\{f_1, f_2, \ldots, f_k\}$ and $F = f_1 + f_2 + \cdots + f_k$. Then by Lemma 3.7 in [11] and Lemma 4.4, we have

$$\sum_{n=1}^{k} \|f_naf_n\| = \sum_{\beta} \|f_\beta(aF)f_\beta\| \leqslant \sum_{n=1}^{\infty} s_n(aF) = \sum_{n=1}^{k} s_n(aF). \tag{4.5}$$

Since $\|F\| = 1$, by Corollary 3.6, $s_n(aF) \leqslant s_n(a)$. Now (4.4) follows easily from (4.5).

The following lemma is a generalization of a result by K. Fan (see [3, p. 48, Lemma 4.2]).

LEMMA 4.6. *Let $a, b \in A$. Then*

$$\sum_{n=1}^{k} s_n(a+b) \leq \sum_{n=1}^{k} s_n(a) + \sum_{n=1}^{k} s_n(b) \qquad (k = 1, 2, \dots). \tag{4.6}$$

PROOF. Write $[a+b] = W^*(a+b)$ and $[a+b] = \Sigma_n s_n(a+b)e_n$ (see (2.2) and (2.4)). Then by Lemma 4.5, we have

$$\begin{aligned}\sum_{n=1}^{k} s_n(a+b) &= \sum_{n=1}^{k} \|e_n[a+b]e_n\| = \sum_{n=1}^{k} \|e_n W^*(a+b)e_n\| \\ &\leq \sum_{n=1}^{k} \|e_n W^* a e_n\| + \sum_{n=1}^{k} \|e_n W^* b e_n\| \\ &\leq \sum_{n=1}^{k} s_n(W^*a) + \sum_{n=1}^{k} s_n(W^*b).\end{aligned} \tag{4.7}$$

Since $\|W^*\| = 1$, (4.6) follows now immediately from (4.7) and Corollary 3.6.

By using Lemma 4.6 and the proof of [3, p. 49, Theorem 4.1], we have

THEOREM 4.7. *Let $a, b \in A$ and $f(x)$ $(0 \leq x < \infty)$ a nondecreasing convex function which vanishes for $x = 0$. Then*

$$\sum_{n=1}^{k} f(s_n(a+b)) \leq \sum_{n=1}^{k} f(s_n(a) + s_n(b)) \qquad (k = 1, 2, \dots).$$

Suppose a is a nonzero element in A with singular values $\{s_n(a)\}$. Define

$$|a|_p = \left(\sum_n s_n^p(a)\right)^{1/p} \qquad (0 < p < \infty)$$

and

$$|a|_\infty = s_1(a).$$

For $a = 0$, we define $|a|_p = 0$ $(0 < p \leq \infty)$. Let $A_p = \{a \in A: |a|_p < \infty\}$ $(0 < p \leq \infty)$. It has been shown that, for $1 \leq p \leq \infty$, A_p is a dual A^*-algebra which is a dense two-sided ideal of A and $A_\infty = A$ (see [11]). We also obtain that A_2 is a proper H^*-algebra with inner product $(\,,)$ such that $(x, x) = |x|_2^2$ and $|ax|_2 \leq \|a\|\,|x|_2$ $(x \in A_2)$. Also, for all $x, y \in A_2$ and $a \in A$, we have $(ax, y) = (x, a^*y)$ and $(xa, y) = (x, ya^*)$. For each $a \in A$, we define a linear operator L_a on A_2 by

$$L_a(x) = ax \qquad (x \in A_2). \tag{4.8}$$

Then L_a is a bounded linear operator on A_2 with $\|L_a\| \leq \|a\|$. Hence $L_a \in B(A_2)$, the algebra of all bounded linear operators on A_2. Clearly $(L_a)^* = L_{a^*}$.

LEMMA 4.8. *Let $a \in A$. Then L_a is a compact operator on A_2 and $s_n(a)$ are the singular values of L_a.*

PROOF. Write $a^*a = \Sigma_\alpha a^*ae_\alpha = \Sigma_n s_n(a)^2 e_n$. Let B be the closure of $\{L_a: a \in A\}$ in $B(A_2)$. Then B is a B^*-algebra. Since $L_{e_\alpha} L_a L_{e_\alpha} = KL_{e_\alpha}$, for some constant k, it follows that L_{e_α} is a Hermitian minimal idempotent in B. Let $F \in B$. If $FL_{e_\alpha} = 0$ for all α, then $FL_{e_\alpha}x = 0$ for all x in A_2. Hence $Fe_\alpha x = 0$. Since by [10, p. 442, Theorem 5.2], $x = \Sigma_\alpha xe_\alpha$ in $|\cdot|_2$, we have $Fx = \Sigma_\alpha Fe_\alpha x = 0$. Consequently $F = 0$ and so $\{L_{e_\alpha}\}$ is a maximal orthogonal family of Hermitian minimal idempotents in B. If M is a closed right ideal of B such that $M \supset \{L_{e_\alpha}\}$, then $M \supset \{L_{e_\alpha b}\}$ for all $b \in A$. Since $L_b = \Sigma_\alpha L_{e_\alpha b}$ in B, it follows that $L_b \in M$ and so $M = B$. Consequently the socle of B is dense in B and so B is a dual B^*-algebra (see [5, p. 20]). Since $L_{a^*a} = \Sigma_n s_n(a)^2 L_{e_n}$, by the proof of Lemma 2.3 in [11], we see that $s_n(a)$ are the singular values of L_a. Since L_{a^*a} is a compact operator on A_2, so is L_a by [6, p. 250, Corollary (4.9.3)] and this completes the proof.

The following lemma is a generalization of a result by A. Horn (see [3, p. 48, Lemma 4.2]).

LEMMA 4.9. *Let $a, b \in A$. Then we have*

$$\prod_{n=1}^{k} s_n(ab) = \prod_{n=1}^{k} s_n(a) \prod_{n=1}^{k} s_n(b) \qquad (k = 1, 2, \ldots). \tag{4.9}$$

PROOF. Write $[ab] = \Sigma_n s_n(ab)e_n$ and let $(\ ,\ )$ be the inner product on A_2. Since by Lemma 3.1 (iv) in [11], $|e_n|_2 = \|e_n\| = 1$, we have

$$(abe_m, abe_n) = ([ab]^2 e_m, e_n) = \begin{cases} s_n^2(ab), & m = n, \\ 0, & m \neq n. \end{cases}$$

Now it follows easily from Lemma 4.8 and [4, p. 375, Theorem 2] that

$$\prod_{n=1}^{k} s_n^2(ab) = \det[(abe_m, abe_n)] \leq \prod_{n=1}^{k} s_n^2(a) \prod_{n=1}^{k} s_n^2(b).$$

Therefore (4.9) holds and this completes the proof.

By using Lemma 4.9 and the proof of [3, p. 49, Theorem 4.2], we can show that

THEOREM 4.10. *Let $a, b \in A$. If the function $f(x)$ $(0 \leq x < \infty; f(0) = 0)$*

becomes convex following the substitution $x = e^t$ $(-\infty \leqslant t < \infty)$, *then we have*

$$\sum_{n=1}^{k} f(s_n(ab)) \leqslant \sum_{n=1}^{k} f(s_n(a)s_n(b)) \qquad (k = 1, 2, \ldots).$$

5. **Symmetric norming functions and symmetrically-normed ideals.** We use the notation in [3, Chapter III]. Let c_0 be the set of all sequences $\xi = \{\xi_j\}_1^\infty$ of real numbers which tend to zero and $\hat{c}$ its subset consisting of all sequences with a finite number of nonzero terms. A real function $\Phi(\xi) = \Phi(\xi_1, \xi_2, \ldots)$, defined on $\hat{c}$ is called a symmetric norming (s.n.) function (or symmetric gauge function) if it has the following properties:

(I) $\Phi(\xi) > 0$ $(\xi \in \hat{c}, \xi \neq 0)$;

(II) for any real constant k, $\Phi(k\xi) = |k|\Phi(\xi)$ $(\xi \in \hat{c})$;

(III) $\Phi(\xi + \eta) \leqslant \Phi(\xi) + \Phi(\eta)$ $(\xi, \eta \in \hat{c})$;

(IV) $\Phi(1, 0, 0, \ldots) = 1$;

(V) $\Phi(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) = (|\xi_{j_1}|, |\xi_{j_2}|, \ldots, |\xi_{j_n}|, 0, 0, \ldots)$,

where $\xi = \{\xi_j\} \in \hat{c}$ and $\{j_1, j_2, \ldots, j_n\}$ is any permutation of integers $1, 2, \ldots, n$. See [3] for some of its properties and an equivalent definition.

As before, A denotes a dual B^*-algebra with norm $\|\cdot\|$. Let A^{**} be its second conjugate space with the Arens product. Then it is well known that A^{**} is a W^*-algebra and A can be identified as a *-subalgebra of A^{**} (e.g. see [7] and [10]). The norm on A^{**} is also denoted by $\|\cdot\|$. By [10, p. 439, Theorem 3.1], A is a two-sided ideal of A^{**}.

We shall make use of the formulas in §§3 and 4 to define and study the symmetrically-normed ideals.

DEFINITION. Let B be a subalgebra of A which contains the socle of A. A norm $|\cdot|$ on B is called a symmetric norm (or uniform crossnorm) if the following conditions are satisfied:

(i) $|e| = 1$ for all Hermitian minimal idempotent e in A.

(ii) If $b \in B$ and $a \in A$ such that $s_j(a) \leqslant ks_j(b)$ for some constant k $(j = 1, 2, \ldots)$, then $a \in B$ and $|a| \leqslant k|b|$.

REMARK 1. B is a two-sided ideal of A^{**}. In fact, let $T \in A^{**}$ and $a \in B$. Since A is a two-sided ideal of A^{**}, $Ta \in A$. By Corollary 3.6, $s_j(Ta) \leqslant \|T\| s_j(a)$. Therefore $Ta \in B$ by (ii). Similarly $aT \in B$. Hence by (ii), we have $|Ta| \leqslant \|T\|\,|a|$ and $|aT| \leqslant \|T\|\,|a|$.

REMARK 2. We have $|\cdot| \geqslant \|\cdot\|$ on B. In fact, let $b \in B$ and e be a Hermitian minimal idempotent. Put $a = s_1(b)e$. Then $s_1(a) = s_1(b)$, $s_2(a) = s_3(a) = \cdots = 0$. Hence by (ii), $|a| \leqslant |b|$. Since by (i) $|a| = |s_1(b)e| = s_1(b) = \|b\|$, it follows that $\|b\| \leqslant |b|$.

REMARK 3. $\|ae\| = |ae|$ and $\|ea\| = |ea|$ for all $a \in A^{**}$ and all Hermitian minimal idempotent e. In fact, $|ae| = |aee| \leqslant \|ae\|\,|e| = \|ae\|$. Hence we have

$|ae| = \|ae\|$. Similarly $|ea| = \|ea\|$. If A is a simple algebra, this property is equivalent to the "cross property" in [8, p. 54, Definition 1 (iv)].

REMARK 4. Since $s_j(a) = s_j(a^*) = s_j([a]) = s_j([a^*])$ by Lemma 3.1 in [11], it follows that B is a $*$-algebra.

REMARK 5. If $a, b \in B$ with $s_j(a) = s_j(b)$ for all j, then $|a| = |b|$.

Let S_A be the socle of A. Then S_A is a two-sided ideal of A^{**}. In the following result, we shall show that the class of all s.n. functions and the class of all symmetric norms on S_A generate each other.

THEOREM 5.1. *If $\Phi(\xi)$ is any s.n. function, then the equality*

$$|a|_\Phi = \Phi(s(a)) \qquad (a \in S_A, s(a) = \{s_j(a)\}) \tag{5.1}$$

defines a symmetric norm on S_A. Conversely, every symmetric norm on S_A is obtained in such a manner.

PROOF. By Lemma 4.6 and [3, p. 75, Lemma 3.2(v′)], we have

$$\begin{aligned}|a+b|_\Phi &\leqslant \Phi(s(a)+s(b)) \leqslant \Phi(s(a)) + \Phi(s(b)) \\ &= |a|_\Phi + |b|_\Phi,\end{aligned}$$

for all a, b in S_A. Since $\Phi(1, 0, 0, \ldots) = 1$, it follows that $|e|_\Phi = 1$ for all Hermitian minimal idempotents e. Property (ii) in Definition easily follows from [3, p. 71, (3.1)]. Therefore (5.1) defines a symmetric norm on S_A.

Conversely, let $|\cdot|$ be a symmetric norm on S_A. Define

$$\Phi(\xi) = \left|\sum_{j=1}^{n} \xi_j e_j\right|, \tag{5.2}$$

where $\xi = \{\xi_j\}_{j=1}^n \in \hat{c}$ and $e_1, e_2, \ldots, e_n$ are mutually orthogonal Hermitian minimal idempotents. It follows easily from Remark 5 that (5.2) is well defined. It is easy to see that $\Phi(\xi)$ is an s.n. function and $|a|_\Phi = |a|$ $(a \in S_A)$. This completes the proof.

REMARK. Some argument in the proof of Theorem 5.1 is similar to that of [8, p. 65, Theorem 5] and [3, p. 78, Theorem 3.1].

Let B be a subalgebra of A with a symmetric norm $|\cdot|$. If B is complete in $|\cdot|$, then it is called a symmetrically-normed (s.n.) ideal.

REMARK. Since B contains the socle of A, B is an A^*-algebra which is a dense two-sided ideal of A.

Let Φ be an s.n. function with the natural domain c_Φ (see [3, p. 80]). For each a in A, let $s(a) = \{s_j(a)\}$. Define

$$A_\Phi = \{a \in A \colon s(a) \in c_\Phi\} \tag{5.3}$$

and

$$|a|_\Phi = \Phi(s(a)). \tag{5.4}$$

REMARK. By using the proof of [3, p. 80, Theorem 4.1], we can show that A_Φ is an s.n. ideal with norm $|a|_\Phi$.

Two s.n. functions $\Phi(\xi)$ and $\Psi(\xi)$ are said to be equivalent if

$$\sup\{\Phi(\xi)/\Psi(\xi)\colon \xi \in \hat{c}\} < \infty \quad \text{and} \quad \sup\{\Psi(\xi)/\Phi(\xi)\colon \xi \in \hat{c}\} < \infty$$

(see [3, p. 76]).

REMARK. It is well known that a semisimple Banach algebra has a unique norm. Therefore two s.n. functions $\Phi(\xi)$ and $\Psi(\xi)$ are equivalent if and only if the s.n. ideal A_Φ and A_Ψ coincide elementwise.

Let $\Phi_\infty(\xi)$ and $\Phi_1(\xi)$ be the minimal and maximal s.n. functions (for definitions, see [3, p. 76]).

REMARK. Since $\|a\| = s_1(a)$, it follows that A_Φ coincides with A if and only if Φ is equivalent to Φ_∞. Since $|a|_1 = \Sigma_{j=1}^\infty s_j(a)$, it follows that A_Φ coincides with A_1 if and only if Φ is equivalent to Φ_1.

As before let S_A be the socle of A. Then A_Φ contains S_A. Let $A_\Phi^{(0)}$ be the closure of S_A in A_Φ. We note that $A_\Phi^{(0)}$ may not be equal to A_Φ (see [3, p. 87]). Clearly $A_\Phi^{(0)}$ is an A^*-algebra which is a dense two-sided ideal of A.

THEOREM 5.2. *$A_\Phi^{(0)}$ is a dual algebra.*

PROOF. Let R_n be given as in §4. For each a in $A_\Phi^{(0)}$, write $[a] = \Sigma_{j=1}^\infty s_j(a)e_j$ and $a_n = \Sigma_{j=1}^n ae_k$. Then $a_n \in R_n$ and by (4.3), $\|a - a_n\| = s_{n+1}(a)$. Therefore by Theorem 4.1, for all T in R_r, we have

$$s_{n+1}(a) = \|(a+T) - (T + a_n)\| \geqslant s_{n+r+1}(a+T). \tag{5.5}$$

Hence by (5.5) and the proof of [3, p. 87, Lemma 6.1], we have

$$\min_{K \in R_n} |a - K|_\Phi = |a - a_n|_\Phi = \Phi(s_{n+1}(a), s_{n+2}(a), \dots). \tag{5.6}$$

Since S_A is dense in $A_\Phi^{(0)}$ and each element of S_A belongs to some R_n, it follows that $a_n \longrightarrow a$ in $|\cdot|_\Phi$. Therefore by [5, p. 29, Lemma 8(3)], $A_\Phi^{(0)}$ is a dual A^*-algebra. This completes the proof.

6. The conjugate space of $A_\Phi^{(0)}$. Let H be a Hilbert space with inner product $(\ ,\ )$. If x and y are elements in H, then $x \otimes y$ will denote the operator on H defined by $(x \otimes y)(h) = (h, y)x$ for all h in H.

LEMMA 6.1. *Let $\{e_\alpha\}$ be a net of Hermitian minimal idempotents in A*

such that $\{e_\alpha\}$ *converges weakly to some element* e *in* A^{**}. *Then* $e \in A$.

Proof. Let $\{I_\lambda\}$ be the set of all closed two-sided minimal ideals of A. We note first that each e_α belongs to some I_λ. Also if $e_{\alpha_1} \in I_{\lambda_1}$ and $e_{\alpha_2} \in I_{\lambda_2}$, then $e_{\alpha_1} e_{\alpha_2} = 0$. We divide the proof into two cases.

Case 1. Suppose each I_λ does not contain any subnet of $\{e_\alpha\}$. Let e_{α_0} be an arbitrary element in $\{e_\alpha\}$. Then e_{α_0} belongs to some I_{λ_0}. Let $\{e_\beta\} = \{e_\alpha\} \cap I_{\lambda_0}$ and $\{e_\gamma\} = \{e_\alpha\} - \{e_\beta\}$. Then $\{e_\gamma\}$ is a subnet of $\{e_\alpha\}$. Since $e_\gamma e_{\alpha_0} = 0$, $ee_{\alpha_0} = 0$. Since e_{α_0} is arbitrary, it follows that $ee_\alpha = 0$ for all α. Therefore $e^2 = 0$. Since $e = e^*$, we have $e = 0$. Hence $e \in A$.

Case 2. Suppose there exists some I_λ which contains a subnet $\{e_\tau\}$ of $\{e_\alpha\}$. It is easy to see that $e \in I_\lambda^{**}$, the second conjugate space of I_λ. Since I_λ is a simple dual B^*-algebra, I_λ has the form $I_\lambda = LC(H_\lambda)$ for some Hilbert space H_λ. Also we can identify $LC(H_\lambda)^{**}$ with $L(H_\lambda)$, the algebra of all continuous linear operators on H_λ. Write $e_\tau = x_\tau \otimes x_\tau$ with $x_\tau \in H_\lambda$. Since $\|x_\tau\| = 1$, we can assume that $\{x_\tau\}$ converges weakly to some x in H_λ. Hence $((x_\tau \otimes x_\tau) y, z) \longrightarrow ((x \otimes x) y, z)$ for all y, z in H. Since $e_\tau \longrightarrow e$ weakly in $LC(H_\lambda)^{**} = L(H_\lambda)$, it follows that $(e_\tau y, z) \longrightarrow (e y, z)$ for all y, z in H. Therefore $e = x \otimes x \in I_\lambda \subset A$. This completes the proof.

The following result is similar to [3, p. 85, Theorem 5.2].

Lemma 6.2. *Let* $\Phi(\xi)$ *be an arbitrary s.n. function not equivalent to the minimal one. If* a *and* a_α *are positive elements in* A^{**} *such that* $a_\alpha \longrightarrow a$ *weakly in* A^{**}, $a_\alpha \in A_\Phi$ *and* $M = \sup_a |a_\alpha|_\Phi < \infty$, *then* $a \in A_\Phi$ *and* $|a|_\Phi \leq M$.

Proof. Let $a_\alpha = \Sigma_{j=1}^\infty s_j(a_\alpha) e_j^{(\alpha)}$ be a spectral representation of a_α in A (see (2.2)). For any fixed positive integer n, let $b_{\alpha,n} = \Sigma_{j=1}^n s_j(a_\alpha) e_j^{(\alpha)}$. Since $s_j(a_\alpha) \leq |a_\alpha|_\Phi \leq M$, there exist subnets $\{a_\beta\}$ and $\{e_j^{(\beta)}\}$ such that $s_j(a_\beta) \longrightarrow s_j$ for some nonnegative number s_j and $e_j^{(\beta)} \longrightarrow e_j$ weakly for some e_j in A^{**}. By Lemma 6.1, $e_j \in A$. Put $b_n = \Sigma_{j=1}^n s_j e_j$ and $K_n = M/\Phi(\underbrace{1, 1, \ldots, 1}_{n}, 0, \ldots)$. Then $b_n \in A$ and

$$\|a_\alpha - b_{\alpha,n}\| = \left\| \sum_{j=n+1}^{\infty} s_j(a_\alpha) e_j^{(\alpha)} \right\| = s_{n+1}(a_\alpha) \leq K_{n+1}.$$

Hence for all f in A^* with $\|f\| \leq 1$, we have

$$|a_\alpha(f) - b_{\alpha,n}(f)| \leq K_{n+1} \quad (n = 1, 2, \ldots). \tag{6.1}$$

Since $a_\beta \longrightarrow a$ and $b_{\beta,n} \longrightarrow b_n$ weakly in A^{**}, by (6.1) we have

$$|(a - b_n)(f)| \leqslant |(a - a_\beta)(f)| + |(a_\beta - b_{\beta,n})(f)| + |(b_{\beta,n} - b_n)(f)|$$
$$\leqslant K_{n+1},$$

for all f in A^* with $\|f\| \leqslant 1$. Therefore $\|a - b_n\| \leqslant K_{n+1}$. Since $K_{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$, it follows that $b_n \longrightarrow a$ and so $a \in A$. Therefore by (2.2), we can write $a = \Sigma_{j=1}^\infty s_j(a) f_j$. Let $t_j = \sup_\alpha s_j(a_\alpha)$ $(j = 1, 2, \ldots)$. Since $\Phi(s_1(a_\alpha), s_2(a_\alpha), \ldots, s_n(a_\alpha), 0, 0, \ldots) \leqslant M$, it follows that

$$\Phi(t_1, t_2, \ldots, t_n, 0, 0, \ldots) \leqslant M.$$

Also by Lemma 4.5, we have

$$\sum_{j=1}^n \|f_j a_\alpha f_j\| \leqslant \sum_{j=1}^n s_j(a_\alpha) \leqslant \sum_{j=1}^n t_j. \tag{6.2}$$

Since $f_j a_\alpha f_j$ is positive, $f_j a_\alpha f_j = \|f_j a_\alpha f_j\| f_j$. Similarly $f_j a f_j = \|f_j a f_j\| f_j$. Since $f_j a_\alpha f_j \longrightarrow f_j a f_j$ weakly in A^{**}, it follows easily that $\|f_j a_\alpha f_j\| \longrightarrow \|f_j a f_j\|$. Hence by (6.2), we have

$$\sum_{j=1}^n s_j(a) = \sum_{j=1}^n \|f_j a f_j\| \leqslant \sum_{j=1}^n t_j$$

and consequently $\Phi(s_1(a), s_2(a), \ldots, s_n(a), 0, 0, \ldots) \leqslant M$. Therefore $a \in A_\Phi$ and this completes the proof.

REMARK. Some argument in the proof of Lemma 6.2 is similar to that given in the proof of [3, p. 85, Theorem 5.2].

Let $a \in A_1$. Then by Theorem 4.3 in [11], $a = c^* b$ for some b, c in A_2. Define

$$\operatorname{tr} a = (b, c) \qquad (a \in A_1), \tag{6.3}$$

where $(\,,)$ denotes the inner product in A_2. Let $\{f_\beta\}$ be a maximal orthogonal family of Hermitian minimal idempotents in A and $\lambda_\beta f_\beta = f_\beta a f_\beta$. Then by Lemma 4.4 in [11], tr a is well defined, $\operatorname{tr} a = \Sigma_\beta (a f_\beta, f_\beta) = \Sigma_\beta \lambda_\beta$ and $|\operatorname{tr} a| \leqslant |a|_1$.

Let Φ be an s.n. function and Φ^* be an s.n. function adjoint to Φ (see [3, p. 125] and [8, p. 69]). By Theorem 4.10 and the proof of [3, p. 49, Corollary 4.1], we have

$$|ax|_1 = \sum_{j=1}^\infty s_j(ax) \leqslant \sum_{j=1}^\infty s_j(a) \sum_{j=1}^\infty s_j(x) \qquad (a, x \in A).$$

Also for all a in A_{Φ^*},

$$|a|_{\Phi^*} = \max_{0 \neq x \in A_\Phi} \left(|x|_\Phi^{-1} \sum_{j=1}^\infty s_j(a) s_j(x) \right).$$

It follows that

(6.4) $$|ax|_1 \leqslant |a|_{\Phi^*}|x|_\Phi \qquad (a \in A_{\Phi^*}, x \in A_\Phi).$$

Let $(A_\Phi^{(0)})^*$ be the conjugate space of $A_\Phi^{(0)}$. We shall show that $(A_\Phi^{(0)})^*$ can be identified with A_{Φ^*}. The following result is a generalization of [3, p. 130, Theorem 12.2].

THEOREM 6.3. *Let $\Phi(\xi)$ be an arbitrary s.n. function, not equivalent to the maximal one. Then for each f in $(A_\Phi^{(0)})^*$, f is of the form*

(6.5) $$f(x) = \operatorname{tr} ax \qquad (x \in A_\Phi^{(0)}),$$

for some a in A_{Φ^*} and $\|f\| = |a|_{\Phi^*}$.

PROOF. Let $a \in A_{\Phi^*}$ and define $f(x) = \operatorname{tr} ax$ ($x \in A_\Phi^{(0)}$). Then by (6.4), we have

$$|f(x)| = |\operatorname{tr} ax| \leqslant |ax|_1 \leqslant |a|_{\Phi^*}|x|_\Phi.$$

Therefore $f \in (A_\Phi^{(0)})^*$ and $\|f\| \leqslant |a|_{\Phi^*}$. To show the converse of the inequality, we put $[a] = \Sigma_{j=1}^\infty s_j(a)e_j$ and $a_n = \Sigma_{j=1}^n ae_j$. Then

$$\Phi^*(s(a_n)) = \Phi^*(s_1(a), s_2(a), \ldots, s_n(a), 0, 0, \ldots).$$

Let $\hat{k}$ be the set of all nonincreasing sequences in $\hat{c}$, each of which consists of nonnegative numbers. Now by the definition of Φ^*, we can choose $\xi^{(n)} = \{\xi_j^{(n)}\}_1^n \in \hat{k}$ such that $\Phi(\xi^{(n)}) = 1$ and $\Sigma_{j=1}^n s_j(a)\xi_j^{(n)} = \Phi^*(s(a_n))$. Put $f_j = s_j(a)^{-2}ae_ja^*$. Then $\{f_j\}$ are mutually orthogonal Hermitian minimal idempotents and $[a^*] = \Sigma_{j=1}^\infty s_j(a)f_j$ (see [11]). Let $b_n = \Sigma_{j=1}^n (\xi_j^{(n)}/s_j(a))a^*f_j$. Then $ab_n = \Sigma_{j=1}^n \xi_j^n s_j(a)f_j$ and so $\operatorname{tr}(ab_n) = \Sigma_{j=1}^n \xi_j^{(n)}s_j(a)$. Since $b_n^*b_n = \Sigma_{j=1}^n(\xi_j^{(n)})^2f_j$, we have $s_j(b_n^*) = \xi_j^{(n)}$ ($j = 1, 2, \ldots, n$). Therefore

$$|b_n|_\Phi = \Phi(s(b_n)) = \Phi(\xi_1^{(n)}, \ldots, \xi_n^{(n)}, 0, 0, \ldots) = 1.$$

Also

$$\begin{aligned} f(b_n) = \operatorname{tr}(ab_n) &= \sum_{j=1}^n \xi_j^{(n)}s_j(a) \\ &= \Phi^*(s_1(a), s_2(a), \ldots, s_n(a), 0, 0, \ldots) \\ &= \Phi^*(s(a_n)) = |a_n|_{\Phi^*}. \end{aligned}$$

Since $|a_n|_{\Phi^*} \longrightarrow |a|_{\Phi^*}$ and $|b_n|_\Phi = 1$, it follows that $\|f\| \geqslant |a|_{\Phi^*}$ and so they are equal.

Conversely let f be a nonzero functional in $(A_\Phi^{(0)})^*$. Since $|\cdot|_\Phi \leqslant |\cdot|_1$ and S_A is dense in $A_\Phi^{(0)}$, it follows that f is a nonzero functional in A_1^*. Hence by Theorem 3.3 in [12], there exists some a in A^{**} such that

(6.6) $$f(x) = \operatorname{tr} ax \qquad (x \in A_1).$$

We show that $a \in A_{\Phi^*}$. In fact, let $\{e_\lambda : \lambda \in \Lambda\}$ be a maximal orthogonal family of Hermitian minimal idempotents in A and let $\{E_\gamma : \gamma \in \Gamma\}$ be the direct set of all finite sums $e_{\lambda_1} + e_{\lambda_2} + \cdots + e_{\lambda_n}$ ($\lambda_n \in \Lambda$ and $n = 1, 2, \ldots$). Then $\|E_\gamma\| = 1$. Define f_γ on $A_\Phi^{(0)}$ by

(6.7) $$f_\gamma(x) = f(xE_\gamma) \qquad (x \in A_\Phi^{(0)}).$$

Then

$$f_\gamma(x) = \operatorname{tr}(axE_\gamma) = \operatorname{tr}(E_\gamma axE_\gamma) = \operatorname{tr}(E_\gamma ax).$$

Since $E_\gamma a \in A_{\Phi^*}$, it follows from the first part of the theorem that $|E_\gamma a|_{\Phi^*} = \|f_\gamma\| \leq \|f\|$. Since A^{**} is a W^*-algebra, by [7, p. 27, Theorem 1.12.1], there exists some $u \in A^{**}$ with $\|u\| = 1$ such that $[a^*] = au$. Therefore, for all γ, we have

$$|E_\gamma [a^*] E_\gamma|_{\Phi^*} = |E_\gamma auE_\gamma|_{\Phi^*} \leq |E_\gamma a|_{\Phi^*} \leq \|f\|.$$

Hence by the Alaoglu theorem, we can assume that $\{E_\gamma [a^*] E_\gamma\}$ converges weakly to some b in A^{**}. Since $\{E_\gamma\}$ converges weakly to the identity in A^{**}, it is easy to see that $(E_\gamma [a^*] E_\gamma)e_\lambda \longrightarrow [a^*] e_\lambda$ and $(E_\gamma [a^*] E_\gamma)e_\lambda \longrightarrow be_\lambda$ weakly in A^{**}. Therefore $be_\lambda = [a^*]e_\lambda$ for all λ and so $b = [a^*]$. Hence $E_\gamma [a^*] E_\gamma \longrightarrow [a^*]$ weakly in A^{**}. Therefore by Lemma 6.2, $[a^*] \in A_{\Phi^*}$ and so is a. This completes the proof.

We remark that some argument in the proof of Theorem 6.3 is similar to that of [3, p. 130, Theorem 12.2].

7. **Some special symmetrically-normed ideals.** Let $\Pi = \{\pi_j\}_1^\infty$ be an arbitrary binormalizing sequence (see [3, p. 141]). Put

$$A_\Pi = \left\{ a \in A : \sup_n \left[\sum_{j=1}^n s_j(a) \Big/ \sum_{j=1}^n \pi_j \right] < \infty \right\}$$

and

$$A_\Pi^{(0)} = \left\{ a \in A : \lim_{n \to \infty} \left[\sum_{j=1}^n s_j(a) \Big/ \sum_{j=1}^n \pi_j \right] = 0 \right\}.$$

THEOREM 7.1. *$A_\Pi^{(0)}$ and A_Π are s.n. ideals of A such that $A_\Pi^{(0)}$ is a proper subspace of A_Π.*

PROOF. This follows from the proof of [3, p. 141, Theorem 14.1].

COROLLARY 7.2. *A_Π is a modular annihilator algebra (for definition, see* [9]), *but not an annihilator algebra.*

PROOF. Since A_Π is a two-sided ideal of A and A is dual, it follows from [9, p. 830, Theorem 5.2] and [9, p. 831, Theorem 5.3] that A_Π is a modular annihilator algebra. Since S_A is the socle of A_Π and S_A is not dense in A_Π, it follows that A_Π is not an annihilator algebra.

Let ϕ_π be defined as in [3, p. 145]. Then ϕ_π is an s.n. function. Let A_π be the s.n. ideal of A obtained from ϕ_π. Then by the proof of [3, p. 149, Theorem 15.2], we have:

THEOREM 7.3. *For the triple of spaces $A_\Pi^{(0)}$, A_π and A_Π, each space is the conjugate space of the preceding one.*

REFERENCES

1. B. A. Barnes, *A generalized Fredholm theory for certain maps in the regular representations of an algebra*, Canad. J. Math. 20 (1968), 495–504. MR 38 #534.

2. N. Dunford and J. T. Schwartz, *Linear operators.* II: *Spectral theory. Self adjoint operators in Hilbert space*, Interscience, New York, 1963. MR 32 #6181.

3. I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, R. I., 1969. MR 36 #3137; 39 #7447.

4. A. Horn, *On the singular values of a product of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 374–375. MR 13, 565.

5. T. Ogasawara and K. Yoshinaga, *Weakly completely continuous Banach *-algebras*, J. Sci. Hiroshima Univ. Ser. A 18 (1954), 15–36. MR 16, 1126.

6. C. E. Rickart, *General theory of Banach algebras*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #5903.

7. S. Sakai, *C^*-algebras and W^*-algebras*, Ergebnisse Math. Grenzgebiete, Band 60, Springer-Verlag, Berlin, 1971.

8. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse Math. Grenzgebiete, N.F., Heft 27, Springer-Verlag, Berlin, 1960. MR 22 #9878.

9. P. K. Wong, *Modular annihilator A^*-algebras*, Pacific J. Math. 37 (1971), 825–834. MR 46 #4213.

10. ———, *On the Arens products and certain Banach algebras*, Trans. Amer. Math. Soc. 180 (1973), 437–448. MR 47 #7431.

11. ———, *The p-class in a dual B^*-algebra*, Trans. Amer. Math. Soc. 200 (1974), 355–368. MR 50 #10837.

12. ———, *On certain subalgebras of a dual B^*-algebra*, J. Australian Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, SETON HALL UNIVERSITY, SOUTH ORANGE, NEW JERSEY 07079